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Ascending HNN extensions of residually finite groups can be non-Hopfian and can have very few finite quotients

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Abstract

We produce two examples demonstrating that an ascending HNN extension of a finitely generated residually finite group need not be residually finite. The first example has very few quotients. The second example is not Hopfian. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A group is *residually finite* provided each non-identity element of this group can be mapped to a non-identity element in some homomorphism onto a finite group. Residually finite groups are important from many points of view. They are natural examples of groups with solvable word problem [14] (see [10, Theorem IV.4.6]). They arise in number theory as Galois groups of certain extensions of the field of rational numbers [7]. These Galois groups were also the first examples of infinite finitely generated torsion groups [2]. Other important examples of residually finite groups are groups generated by automata (groups discovered by Aleshin, Suschanskii, Grigorchuk, Gupta, Sidki and others) and many other groups acting on rooted trees. Finitely generated matrix groups over any commutative ring are residually finite [12], and consequently, so are free groups, polycyclic groups, finitely generated metabelian groups etc., because they are representable by matrices.

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Many constructions preserve residual finiteness. For example, the cartesian and free products of residually finite groups are residually finite [6], and the split extension of a finitely generated residually finite group by a residually finite group is residually finite [13]. In particular, the HNN extension of every residually finite group via an automorphism is residually finite because it is simply a semidirect product of the residually finite group and a cyclic group.

A beautiful result of Malcev states that every finitely generated residually finite group is *Hopfian* which means that it is not isomorphic to a proper quotient of itself [12] (see [10, Theorem IV.4.10]). It is often easier to check that a group is not Hopfian than to prove directly that it is not residually finite. Indeed, to show that a group is not Hopfian, it is enough to manufacture one particular “bad” endomorphism of this group which is surjective but not injective.

Thus a statement that a group is Hopfian can be considered an “approximation” of the statement that the group is residually finite. For example, Sela proved [17] that torsion-free word-hyperbolic groups are Hopfian. Note that it is still an open question whether every word-hyperbolic group is residually finite [5].

Both residual finiteness and the Hopf property are relevant in topology. For example, the fundamental group of a manifold M is residually finite iff for every essential closed path σ in M there is a finite cover \hat{M} such that σ does not lift to a closed path in \hat{M} . The original interest in the Hopf property grew out of Hopf’s consideration in 1931 of the following topological problem: Is every degree 1 map $\phi : M \rightarrow M$ of a closed manifold a homotopy equivalence? When M is aspherical, Hopf observed that this is equivalent to the problem of whether $\pi_1 M$ is Hopfian, and he asked whether every finitely generated group is Hopfian.

It took a long time before the first finitely generated non-Hopfian group was constructed by Neumann. Shortly thereafter, Higman discovered [9] the following non-Hopfian finitely presented group: $\langle a, s, t \mid a^s = a^2, a^t = a^2 \rangle$. (The endomorphism induced by $a \mapsto a^2, s \mapsto s, t \mapsto t$ is surjective but not injective.)

Higman’s example has been generalized in various ways. One famous example [1] is the Baumslag-Solitar group $\langle b, s \mid (b^2)^t = b^3 \rangle$. (The endomorphism induced by $b \mapsto b^2, s \mapsto s$ is surjective but not injective.)

Another generalization of Higman’s example was given by Meier in [15]. First consider an injective endomorphism ϕ of a group H , such that $\phi(H) \neq H$ (such a group H is said to fail to be *co-Hopfian*). Next form the multiple HNN extension $\langle H, s, t \mid h^s = \phi(h), h^t = \phi(h) \rangle$. As in Higman’s example the resulting group G fails to be Hopfian because of the endomorphism $\Phi : G \rightarrow G$ induced by $\phi : H \rightarrow H$ and $s, t \mapsto s, t$.

More recently, Higman’s idea was refined further in [19] by using the observation that the stable letters need not conjugate the *entire* group H to itself using ϕ . Rather, the stable letters can conjugate proper subgroups of H in a manner which commutes with ϕ . For instance, let $H = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a, b] \rangle$, and let ϕ be induced by $a \mapsto a^2, b \mapsto b^2$. Then the multiple HNN extension $\langle a, b, s, t \mid [a, b], a^s = b^2, b^t = (aab)^2 \rangle$ is not Hopfian. (The endomorphism induced by $a, b, s, t \mapsto a^2, b^2, s, t$ is surjective but not injective.)

Many other non-Hopfian groups can be constructed in this way. In particular, this was used in [19] to produce non-Hopfian groups which are the fundamental groups of compact non-positively curved 2-complexes.

Let $\phi: H \rightarrow H$ be an injective endomorphism. The *ascending* HNN extension of H determined by ϕ is defined to be the HNN extension $\langle H, t \mid h^t = \phi(h): \forall h \in H \rangle$. The ascending HNN extension is sometimes called a *descending* HNN extension, and sometimes called the *mapping torus* of ϕ .

The main result of this paper is that:

Theorem 1.1. *There is an ascending HNN extension of a f.g. residually finite group which is not Hopfian.*

No such example was previously known to exist. Theorem 1.1 is the subject of Section 3 where this example is constructed in Example 3.5. Since f.g. non-Hopfian groups are not residually finite, Theorem 1.1 also provides:

Corollary 1.2. *There is an ascending HNN extension of a f.g. residually finite group which is not residually finite.*

Corollary 1.2 is amplified by an example of an ascending HNN extension of a finitely generated residually finite group, which has very few finite quotients. This example is the subject of Section 2.

We note that there are some cases where ascending HNN extensions of f.g. residually finite groups are residually finite. For instance, Moldavanskiĭ [16] showed that ascending HNN extensions of virtually nilpotent groups are residually finite.

Corollary 1.2 is particularly interesting in view of the following result of Malcev [13] that was mentioned above: If G can be expressed as a split extension $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ where N is f.g. residually finite, and Q is residually finite, then G is residually finite. In particular, if G is an extension of a f.g. residually finite group by a cyclic group, then G is residually finite. Of course, an extension of a group by an infinite cyclic group, is an ascending HNN extension which is not *properly* ascending.

We close this section with the following conjecture which can perhaps only be supported by the sentiment that word-hyperbolic groups are well-behaved.

Conjecture 1.3. *Let H be a (residually finite) word-hyperbolic group, and let $\phi: H \rightarrow H$ be an injective endomorphism of H , then the resulting ascending HNN extension is (residually finite) Hopfian.*

2. An example with few quotients

We first present an idea for constructing a non-residually finite ascending HNN extension of a residually finite group. This idea did not quite work but it led to a correct example.

Lemma 2.1. *Let G be a nontrivial group $\langle x_1, x_2, \dots, x_n \rangle$ which does not have a homomorphism onto any finite non-abelian simple group. Furthermore, suppose that G contains an isomorphic copy of itself inside its derived subgroup, with corresponding generators $\langle y_1, \dots, y_n \rangle$. Then every finite quotient of the ascending HNN extension $\tilde{G} = \langle x_1, \dots, x_n, t \mid x_i^t = y_i \rangle$ is cyclic. In particular, G is not residually finite.*

Proof. To prove this, first observe that if f is a homomorphism from \tilde{G} onto a finite group, then $f(t)$ conjugates $f(G)$ into a subgroup of its derived subgroup, and therefore since $f(G)$ is finite, it must equal its derived subgroup and is therefore perfect.

On the other hand, by hypothesis, G and therefore $f(G)$ has no non-abelian simple quotient. Since a nontrivial perfect group has a non-abelian simple quotient, we conclude that $f(G)$ must be trivial. Consequently G is mapped to the identity in any finite quotient of \tilde{G} , and therefore any finite quotient of \tilde{G} is a cyclic group generated by the image of t . \square

It is easy to construct (even finitely presented) residually finite groups without non-abelian simple finite quotients. For example, most free by cyclic groups have this property. But in order to apply Lemma 2.1, one would also need this group to contain a copy of itself in its derived subgroup. This condition is much harder to satisfy and we do not know any examples of such groups. Thus we formulate the following problem.

Problem 2.2. *Is there a finitely presented residually finite group without finite simple non-abelian quotients which contains an isomorphic copy of itself in its derived subgroup?*

Lemma 2.1 suggests the consideration of finitely generated residually finite p -groups, because such groups certainly cannot have homomorphisms onto finite non-abelian simple groups. One of the many known examples of such groups is the Grigorchuk group G [3]. In addition to showing that his group has intermediate growth, Grigorchuk proved that every proper quotient of G is a finite 2-group.

Lysenok [11] gave the following presentation of Grigorchuk's group:

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1, U_k = V_k = 1: (k \geq 0) \rangle$$

where $U_0 = (ad)^4$, $V_0 = (adacac)^4$ and $U_k = \sigma^k(U_0)$, $V_k = \sigma^k(V_0)$ for $k \geq 1$, where σ is induced by $a \mapsto aca$, $b \mapsto d$, $c \mapsto b$, $d \mapsto c$.

We do not know if the Grigorchuk group contains a copy of itself in its derived subgroup G' , and since G' has finite index in G , there is a chance that G' does contain a copy of G . Nevertheless, there exists an ascending HNN extension of G which is not residually finite, and the proof of this substantially follows the lines of the proof of Lemma 2.1.

Note that the map σ in Lysenok's presentation of the group G is an injective but not surjective endomorphism of G .

The corresponding HNN extension \tilde{G} of G was studied by Grigorchuk who proved that \tilde{G} is amenable but not elementary amenable [3]. It turns out that \tilde{G} has the following finite presentation:

$$\langle a, b, c, d, t \mid a^2, b^2, bcd, (ad)^4, (adacac)^4, a^t = aca, b^t = d, c^t = b, d^t = c \rangle.$$

Although G is residually finite, the HNN extension \tilde{G} is not.

Theorem 2.3. *Every proper homomorphic image of \tilde{G} is metabelian and has an abelian subgroup of finite index.*

Proof. Indeed, let H be a proper homomorphic image of \tilde{G} under a homomorphism ϕ . We will denote the images of a, b, c, d, t by $\alpha, \beta, \gamma, \delta, \tau$. Consider the subgroup $\phi(G)$ of H , we will first show that $\phi(G)$ is a proper quotient of Grigorchuk's group G . Every element of \tilde{G} can be represented in the form $t^n g t^{-m}$ for some $n, m \geq 0$, $g \in G$. This element equals 1 if and only if $n = m$ and $g = 1$. The homomorphism ϕ has a non-identity element $z = t^n g t^{-m}$ in its kernel. If $n = m$ then $g \neq 1$ is in the kernel of ϕ and so $\phi(G)$ is a proper homomorphic image of G . If $n \neq m$ then $\phi(t^{m-n}) = \phi(g)$. Without loss of generality we can assume that $m > n$. Since σ is not surjective there exists an element s in G such that $s \notin \sigma(G)$. Therefore $s \notin G^t \subset \tilde{G}$. By the Normal Form Theorem for HNN Extensions [10, Theorem IV.2.1], $(s^{t^{m-n}})^{g^{-1}}$ and s are distinct elements of G . On the other hand it is easy to see that the images of these elements under ϕ coincide. Therefore in this case too, the kernel of ϕ intersects G nontrivially and $\phi(G)$ is a proper quotient of G .

It is proven in [3] that any proper quotient of G is a finite 2-group, and so $\phi(G)$ is a finite 2-group and therefore nilpotent. Observe that τ conjugates $\phi(G)$ into itself and therefore, since $\phi(G)$ is finite, τ conjugates $\phi(G)$ onto $\phi(G)$, and so $\phi(G)$ is normal.

We will now show that $\phi(G)$ is abelian. First note that because of the relations $bcd = bb = cc = dd = 1$, the subgroup $\langle b, c, d \rangle$ of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and in particular it is abelian. Let x denote $[\gamma, \alpha]$. We have $x^\tau = [\beta, \alpha\gamma\alpha] = [\beta, \gamma[\gamma, \alpha]]$, and since β and γ commute, $x^\tau = [\beta, [\gamma, \alpha]]$. Continuing in this way: $x^{\tau^2} = [\delta, [\beta, [\gamma, \alpha]]]$, and so on, we see that x^{t^n} belongs to the n th member of the lower central series of $\phi(G)$. Since $\phi(G)$ is nilpotent, $x^{\tau^n} = 1$ for some n , and so $[\gamma, \alpha] = x = 1$. A similar argument shows that $[\beta, \alpha] = 1$, and $[\delta, \alpha] = 1$, and so $\phi(G)$ is abelian. Since the four generators of $\phi(G)$ are involutions, we see that $\phi(G)$ is an elementary abelian group. In fact, since $\langle \alpha, \beta, \gamma \rangle$ is a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and since the relations $a^t = aca, d^t = c$ imply that $\alpha = \delta$, we see that $\phi(G)$ is a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore H contains an elementary abelian normal subgroup of order at most 4 with a cyclic factor-group. So H is virtually cyclic and metabelian. \square

In [4], it is incorrectly stated (without a proof) that \tilde{G} has no non-cyclic proper quotient. We now describe a non-cyclic quotient of \tilde{G} which is a semidirect product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and \mathbb{Z} defined as follows: First observe that there is an automorphism ψ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ which cycles the nontrivial elements as follows: $(1, 0) \rightarrow (1, 1) \rightarrow (0, 1) \rightarrow$

$(1, 0)$. Now, form the semidirect product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\psi} \mathbb{Z}$ defined by letting conjugation by the generator of \mathbb{Z} induce the automorphism ψ . The ascending HNN extension of G surjects onto this semidirect product by an obvious map.

3. Non-Hopfian example

In this section, we describe an example T of a non-Hopfian group which is an ascending HNN extension of a finitely generated residually finite group G . The base group G is a variation of a peculiar f.g. residually finite group constructed in [20]. The group T is described in Example 3.5, after several lemmas and a definition which will be used to explain its structure and properties. Finally, a finite presentation for T is given in Remark 3.8.

The idea for constructing a non-Hopfian mapping torus of an injective endomorphism is contained in the following lemma.

Lemma 3.1. *Let G be a group. Let $\rho : G \rightarrow G$ be an injective homomorphism. Let $\phi : G \rightarrow G$ be a homomorphism which is not injective. Suppose that ρ and ϕ commute, that is, $\rho \circ \phi = \phi \circ \rho$. And suppose that $\rho(G) \subset \phi(G)$. Let T be the mapping torus of ρ , so $T = \langle G, t \mid g^t = \rho(g) \rangle$. Then T is not Hopfian.*

Proof. Extend $\phi : G \rightarrow G$ to $\phi : T \rightarrow T$ by defining $\phi(t) = t$. Observe that this is a homomorphism because the relations of T are mapped to relations of T . That is, $g^t = \rho(g)$ is mapped to $\phi(g)^t = \phi(\rho(g))$ and since ϕ and ρ commute this is the same as $\phi(g)^t = \rho(\phi(g))$ which is a relation of T .

We now show that $\phi : T \rightarrow T$ is surjective but not injective, and thus T is not Hopfian. To see that $\phi : T \rightarrow T$ is surjective note that t is obviously in the image, and since $t\phi(G)t^{-1} = \rho^{-1}(\phi(G)) = G$, we see that G is in the image as well.

Since $\phi : G \rightarrow G$ is not injective, obviously $\phi : T \rightarrow T$ is not injective. \square

Definition 3.2. Let $H \subset F$ be a subgroup of a group. Let \tilde{F} be an isomorphic copy of F , and let \tilde{H} be the copy of H in \tilde{F} . By the *double of F along H* , we mean the amalgamated free product $G = F *_{{H=\tilde{H}}} \tilde{F}$ where we amalgamate the two copies of F by identifying H with \tilde{H} .

Our group G will be a double, and we will use Lemma 3.3 to see that G is residually finite. Its proof is easy and can be found for instance, in [20]. Recall that a subgroup $H \subset F$ is *closed* provided that it is the intersection of finite index subgroups of F .

Lemma 3.3. *Let H be a closed subgroup of the residually finite group F , then the double $F *_{{H=\tilde{H}}} \tilde{F}$ is residually finite.*

The following lemma states two useful facts about the extension of an endomorphism of F to an endomorphism of a double of F .

Lemma 3.4. *Let G be the double of F along H , and let $\psi : F \rightarrow F$ be an endomorphism such that $\psi(H) \subset H$. Observe that the homomorphism $\psi : F \rightarrow F$ induces a homomorphism $\psi^* : G \rightarrow G$.*

1. ψ^* is injective if and only if ψ is injective and $\psi^{-1}(H) = H$.
2. If ρ and ϕ are endomorphisms of F which map H into itself, then $\rho^* \circ \phi^* = \phi^* \circ \rho^*$ if and only if $\rho \circ \phi = \phi \circ \rho$.

Proof. Statement 1 follows from the Normal Form Theorem for amalgamated free products [10, Theorem IV.2.6]. Statement 2 follows from the commutativity on each factor. \square

Example 3.5. By combining Lemmas 3.1, 3.3, and 3.4, we see that it is sufficient to produce:

1. A closed subgroup H of a f.g. residually finite group F ,
2. Endomorphisms ρ and ϕ of F such that ρ is injective and $\phi \circ \rho = \rho \circ \phi$,
3. $\phi(H) \subset H$ and $\rho(H) \subset H$,
4. $\phi^{-1}(H) \neq H$ and $\rho^{-1}(H) = H$, and
5. $\rho(F) \subset \phi(F)$.

Let $G = F *_H \bar{F}$ be the double of F along H . Then the mapping torus of the injection $\rho^* : G \rightarrow G$ is not Hopfian by Lemma 3.1. Indeed, ρ^* and ϕ^* are respectively injective and non-injective, and ρ^* and ϕ^* commute, and $\rho^*(G) \subset \phi^*(G)$. Consequently, ϕ^* extends to an endomorphism of the mapping torus of ρ^* which is surjective but not injective. Since F is residually finite, and H is closed, we see that G is residually finite by Lemma 3.3.

We now choose $H \subset F$, ρ , and ϕ .

Let $F = \langle a, b, c \rangle$ be free of rank 3. Let $H = \langle c^n b^{(2^n)} a^{(2^n)} b^{-(2^n)} c^{-n} : n \geq 0 \rangle$.

Let $\phi : F \rightarrow F$ be induced by $a \mapsto a^2$, $b \mapsto b$, and $c \mapsto c$. Let $\rho : F \rightarrow F$ be induced by $a \mapsto ca^2c^{-1}$, $b \mapsto cb^2c^{-1}$, and $c \mapsto c$.

One sees that $\rho \circ \phi = \phi \circ \rho$ by checking it for the generators of F . Obviously $\phi(H) \subset H$ because ϕ just squares the generators of H , however we will now show that $\phi^{-1}(H) \neq H$. Indeed $cb^2ab^{-2}c^{-1} \in \phi^{-1}(H)$ because ϕ maps it to a generator of H . However $cb^2ab^{-2}c^{-1} \notin H$ because the corresponding path is not contained in the core illustrated in Fig. 1 (see Remark 3.6 below for a brief discussion of cores).

It is clear that $\rho(H) \subset H$ because ρ just shifts the generators of H . To see that $\rho^{-1}(H) = H$, we first observe that since ρ is injective, it suffices to show that for each $g \notin H$ we have $\rho(g) \notin H - \rho(H)$. Any reduced word W in $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}$ representing an element of $H - \rho(H)$, can be expressed as $W = Uba^n b^{-1}V$ where U does not end with $b^{\pm 1}$ and V does not begin with $b^{\pm 1}$ (but U and/or V might be empty). To see this, observe that W represents a closed immersed based path in the based core of the covering space \hat{X} corresponding to H . (See Remark 3.6 for a brief discussion of cores, and see Fig. 1 for a description of the core of \hat{X} .) The core of the based cover corresponding to $\rho(H)$ can be obtained from the core of \hat{X} by removing the first lollipop (which corresponds to the omitted generator). Consequently, since W

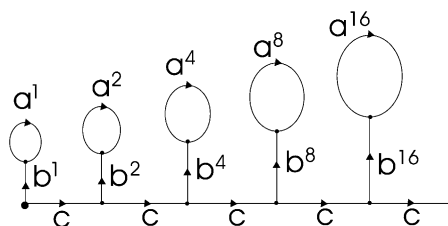


Fig. 1. Above is a partial illustration of the core of \hat{X} which corresponds to the subgroup H . The bold vertex is the basepoint. The core of \hat{X} consists of an infinite ray c^∞ beginning at the basepoint, and consisting of c -edges. Beginning at the basepoint, there is a sequence of lollipops attached at the vertices of c^∞ . These lollipops are growing exponentially (the figure is not drawn to scale). The i th lollipop consists of a stick which is an arc of $2^i b$ edges and a circle of $2^i a$ edges at the top of the stick.

represents an element of $H - \rho(H)$, this path must pass through the first lollipop at the basepoint of \hat{X} . A subpath of W passing through this lollipop provides the subword $ba^n b^{-1}$ of W with the desired property. In particular, W must have a b syllable of odd length. However, since ρ sends any word Z to a word $\rho(Z)$ where each b syllable of $\rho(Z)$ has even length, we see that $\rho(Z)$ cannot be freely equivalent to a word W representing an element of $H - \rho(H)$.

Finally, we will prove that H is closed in Lemma 3.7.

Remark 3.6 (Based cores). We briefly recall some useful notions developed in [18]. Let B denote a bouquet of labeled oriented circles. An edge-path in B is said to be an *immersed* path provided that it has no backtracks. The *based core* of B is the smallest subgraph of B which contains all immersed paths in B which begin and end at b . Note that B can be reconstructed from its based core, by gluing trees to vertices of the core. The case of greatest interest in this paper is when X is a bouquet of labeled oriented circles, and \hat{X} is a based covering space of X . The based core of \hat{X} is then a based labeled oriented graph, and is sometimes called an automaton. If we identify $\pi_1 X$ with the free group F on a set of generators corresponding to the oriented labeled circles of X , then any reduced word in the generators of F corresponds to an immersed based edge-path in X .

When \hat{X} is a covering space corresponding to a finitely generated subgroup of $\pi_1 X$, then the based core of \hat{X} can be constructed by hand as follows: Let W_1, \dots, W_r be a set of based edge-paths in X which generated $\pi_1 \hat{X}$. We wedge together a set of labeled subdivided circles corresponding to the W_i . We then systematically fold together pairs of edges at a vertex, which have the same incoming or outgoing labels. After folding finitely many times, we reach a based labeled graph which is the core of \hat{X} . We refer to [18] for a methodical treatment of these matters. In every example arising in this paper, the reader can verify that a based labeled graph Y is the core of a cover \hat{X} , by checking that the set of generators W_1, W_2, \dots for $\pi_1 \hat{X}$, is a set of generators for $\pi_1 Y$.

Lemma 3.7. *The subgroup $H \subset F$ described above is a closed subgroup of F .*

Proof. Since by [8], each f.g. subgroup of a free group is the intersection of finite index subgroups, it is sufficient to show that H is the intersection of a sequence of finitely generated subgroups of F . Specifically, we show that $H = \bigcap_{n \geq 1} H_n$ where for each $n \geq 1$ the subgroup $H_n \subset F$ is generated as follows:

$$H_n = \langle c^j b^{(2^j)} a^{(2^j)} b^{-(2^j)} c^{-j}, c^j b^{(2^n)} c^{-j}, c^j a^{(2^n)} c^{-j}, c^n : 0 \leq j < n \rangle.$$

To see that $H \subset H_n$ for each n , we show that each generator of H is an element of H_n . So for each $i \geq 0$ we have a generator $c^i b^{(2^i)} a^{(2^i)} b^{-(2^i)} c^{-i}$, and if $i < n$, then the generator is actually one of the generators of H_n . Otherwise, letting $i = rn + q$ where $0 \leq q < n$, we see that

$$\begin{aligned} c^i b^{(2^i)} a^{(2^i)} b^{-(2^i)} c^{-i} &= c^{rn+q} b^{(2^{rn+q})} a^{(2^{rn+q})} b^{-(2^{rn+q})} c^{-(rn+q)} \\ &= (c^n)^r (c^q b^{2^q} c^{-q})^{2^{rn}} (c^q a^{2^q} c^{-q})^{2^{rn}} (c^q b^{2^q} c^{-q})^{-2^{rn}} (c^n)^{-r}. \end{aligned}$$

Since $0 \leq q < n$, the elements in parenthesis on the right-hand side of the equation are generators of H_n , and we have expressed the generator of H as the product of generators of H_n .

We must now show that for each element $g \notin H$, there exists n such that $g \notin H_n$. To prove this, we employ the notion of a based core discussed in Remark 3.6. We identify F with $\pi_1 X$ where X is a bouquet of 3 circles, whose edges are directed and labeled by a , b , and c . Let \hat{X} and \hat{X}_n denote the based covers of X corresponding to H and H_n . Observe that the core Y of \hat{X} consists of a positive c ray which we denote by c^∞ and which begins at the basepoint. For $i \geq 0$, attached along the i th vertex of this ray is a *lollipop* consisting of a *stick* which is an arc of 2^i b edges, with a *circle* of 2^i a edges attached at its end. We refer the reader to Fig. 1 for a partial illustration of the core of \hat{X} .

The core Y_n of \hat{X}_n is more complicated. It consists of a c^n *circle* at the basepoint, together with various other labeled graphs attached to it: For $0 \leq i < n$, there is a 2^i lollipop as above, attached at the i th vertex of the c^n circle, where the i th vertex is the endpoint of the positive path labeled by c^i originating at the basepoint. For $0 \leq i < n$, there is an a^{2^n} circle attached at the i th vertex, and we shall refer to each such circle as an a^{2^n} *ring*. And finally, for $0 \leq i < n$, the stick of each lollipop labeled by b^{2^i} is completed to a b^{2^n} circle by the addition of an arc labeled $b^{2^n-2^i}$ which we shall refer to as a *stem*. We refer the reader to Fig. 2 for an illustration of the based core of \hat{X}_4 .

It is sufficient to show that if w is a reduced word and $w \notin H$ then $w \notin H_n$ for $n > 2|w|$. Equivalently, suppose that $n > 2|w|$ and w is a reduced word with $w \in H_n$ then we will show $w \in H$. Let p denote the closed based path in \hat{X}_n corresponding to w and note that p is a closed based path in Y_n . Since w is a reduced word, the path p is an immersion, and consequently p cannot enter any edge in any of the a^{2^n} rings attached to the c^n circle of Y . Indeed, p cannot backtrack because it is an immersed path, and p is not long enough to pass through the entire circle, and so p would have to terminate in the interior of the circle which is impossible because p is closed. The same reasoning shows that p cannot enter any b^m arc where $m > |w|$. In particular, for $0 \leq i < n$, the $b^{2^n-2^i}$ stem attached alongside the b^{2^i} stick of the i th lollipop cannot be

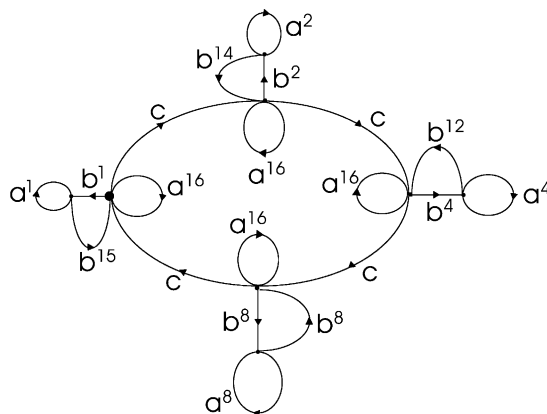


Fig. 2. Above is an illustration of the core of \hat{X}_4 which corresponds to the subgroup H_4 . At the bold basepoint of \hat{X}_4 there is a circle of 4 c edges. Beginning at the basepoint, there is a sequence of lollipops of size 1, 2, 4, 8 attached at the vertices of the c^4 circle. The stick of each lollipop is completed to a b^{2^i} circle by the addition of a b stem. Finally, attached at each vertex of the c^4 circle, there is a circle of 2^4 a edges called a ring.

entered by p , and for $n/2 \leq i < n$, the b^{2^i} stick of the i th lollipop cannot be entered by p . Let Y'_n denote the based component of the graph obtained from Y_n by removing the edges in these a rings, b sticks, and b stems, and observe that p is a closed path in Y'_n .

Let D denote the subgraph of Y'_n which consists of the part of the c^n circle that is traced by the path $c^{-(|w|+1)}$ originating at the basepoint. (This path runs counterclockwise from the basepoint in the depiction of \hat{X}_n in Fig. 2.) The interior vertices of D are the i th vertices of the c^n circle, where $2^n - |w| \leq i < 2^n$. But since $n/2 < 2^n - |w|$, we see that except at the endpoints of D , the vertices of D have no a edges or b edges incident at them because all such a edges and b edges were removed when we obtained Y'_n from Y_n . Now, since D has length $|w| + 1$ which is greater than the length of p , we see that if the path p entered D , then since p cannot backtrack, p would terminate in the interior of D which is impossible because p is closed. We conclude that p is a closed path in the subgraph Y''_n of Y'_n which is obtained by removing D .

Finally, we observe that Y''_n is formed from its c ray, by adding appropriately sized lollipops, and therefore Y''_n is a based subgraph of the based core Y of \hat{X} (Fig. 1). Consequently, p is a closed based path in \hat{X} , and so $w \in H$. \square

Remark 3.8. The double G of F has infinitely generated second homology and therefore cannot be finitely presented. However we note that the ascending HNN extension T of G is finitely presented. If we use A, B, C to denote the basis for \bar{F} , then

$$G = \langle a, b, c, A, B, C \mid c^n b^{2^n} a^{2^n} b^{-2^n} c^{-n} = C^n B^{2^n} A^{2^n} B^{-2^n} C^{-n}; n \geq 0 \rangle.$$

And using the notation $x^y = y^{-1}xy$, we have the presentation:

$$T = \left\langle a, b, c, A, B, C, t \left| \begin{array}{l} c^n b^{2^n} a^{2^n} b^{-2^n} c^{-n} = C^n B^{2^n} A^{2^n} B^{-2^n} C^{-n}; n \geq 0, \\ a^t = ca^2 c^{-1}, \quad b^t = cb^2 c^{-1}, \quad c^t = c, \\ A^t = CA^2 C^{-1}, \quad B^t = CB^2 C^{-1}, \quad C^t = C \end{array} \right. \right\rangle.$$

But since the infinite family of relators of G that appear in this presentation of T are conjugates of the first relator of G by powers of t , we see that:

$$T = \left\langle a, b, c, A, B, C, t \left| \begin{array}{l} bab^{-1} = BAB^{-1}, \\ a^t = ca^2 c^{-1}, \quad b^t = cb^2 c^{-1}, \quad c^t = c, \\ A^t = CA^2 C^{-1}, \quad B^t = CB^2 C^{-1}, \quad C^t = C \end{array} \right. \right\rangle.$$

Since the base groups of both HNN extensions considered in this paper are not finitely presented, it is natural to pose the following problem.

Problem 3.9. Is there a non-residually finite ascending HNN extension of a finitely presented residually finite group?

By Lemma 2.1, a positive solution to Problem 2.2 would provide a positive solution to Problem 3.9. But one may also approach Problem 3.9 using the method of Section 3.

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